Complex Elements in Geometry.

The Theory of the Imaginary in Geometry, together with the Trigonometry of the Imaginary. By Prof. J. L. S. Hatton. Pp. vii+215. (Cambridge: At the University Press, 1920.) Price 18*s.* net.

When we interpret $\phi(x, y)=0$, $\psi(x, y)=0$ as the point-equations of two loci, we are bound to consider any values (x_i, y_i) which satisfy both equations as the co-ordinates of a point common to both curves. The simplest case is when ϕ, ψ are polynomials with ordinary integral coefficients; here the values (x_i, y_i) are determinate, and can be calculated, either exactly or to any desired degree of approximation. Abstractly, (x_i, y_i) are a perfectly definite set of couples of algebraic numbers. A couple (x_i, y_i) may be real, and then corresponds to a real point; but it may be, and often is, complex. What is the most appropriate and fruitful way, from a geometrical point of view, of interpreting these complex solutions of the given pair of equations? This is one of the fundamental problems of analytical geometry, and there are two ways in which it may be attacked. Suppose that the coefficients of ϕ, ψ are real, complex intersections (x_i, y_i) fall into conjugate pairs. The usual analytical formula gives a real line as the join of two conjugate points, and we may call this a common chord of the two loci. The visible result of combining $\phi=0$, $\psi=0$ may be said to be a certain number of real intersections and a certain number of real lines which, from an algebraical point of view, are to be regarded as common chords. The most familiar case is that of two circles and their radical axis; and here we have a geometrical definition of the radical axis which applies whether it meets the two circles or not. We can construct a definition of a common chord of two conics by analogy, whether it meets them in two real or two conjugate complex points; but the procedure is artificial, and there is no obvious way of extending it to higher curves.

The other way is to try to find, as the image or representative of (x, y), when x, y are not both real, some definite constructible geometrical entity to which we can give the name of "point" *without violating the axioms of projective geometry* —*e.g.* it must still be true that any two points determine a line, and so on. This, of course, involves an appropriate definition of a complex line.

It is to von Staudt that we owe an absolutely perfect solution of this difficult problem. Its basic idea is this: Given a real conic, and a real line which does not cut it (in the ordinary sense), there is on the line an elliptic involution of pairs of points conjugate to the conic. With this elliptic involution we can associate either of two opposite "senses" (or directions), and we can interpret the involution, with either sense, as a complex point. These complex points are distinct, and conjugate in a sense analogous to the algebraic one. This geometrical distinction of conjugate complex points appears to have been the one thing with which von Staudt had the greatest difficulty; it must be remembered that he was trying to find a theory applicable to three dimensions as well as to two, and that he wanted to define the line joining *any* two points in space whether real or complex, and this by purely projective considerations. The "join" of two non-conjugate complex points in space is von Staudt's "line of the second kind," and the most difficult to realise of all his concepts.

What we may call a metrical, or Cartesian, image of a complex point $(a+bi, c+di)$ is a segment OP drawn from the real point (a, c) to the real point $(a+b, c+d)$. The conjugate point is represented by a segment $\overrightarrow{OP'}=-\overrightarrow{OP}$, and these two conjugate points are on the real line PP′. Poncelet, following that Will-o'-the-Wisp, the "principle of continuity," very nearly hit upon this representation; for if we consider $x^2+y^2=a^2$, $x=b$ $(b>a)$, we have as the intersections $(b, \pm i\sqrt{b^2-a^2})$, which, in this representation, are the principal ordinates of the real hyperbola $x^2-y^2=a^2$.

Prof. Hatton practically adopts this metrical definition, but in doing so, as it seems to us, introduces unnecessary vagueness, and occasionally wabbles between the two points of view. He begins by an "axiom" which von Staudt breaks up into two definitions, and, so far as we can see, ignores it in all his algebraical "verifications." There is no such thing as an algebraical verification in the true theory. The algebra is taken for granted, and we have to show that our geometrical definitions and postulates and axioms agree with ordinary complex algebra. In the Cartesian representation a point which we may call $(\overrightarrow{OP})$, or more simply (OP), corresponds to von Staudt's representation (O ∞PP′), where O bisects PP′, and ∞ is the point at infinity on the real line POP′.

So long as we keep to von Staudt's projective definition, the questions of such things as "distance," "angle," etc., do not arise. "Sense" and "order" are essential, the latter especially when we consider von Staudt's theory of "casts" and cross-ratios.

It is in connection with the Cartesian imagery

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that we are confronted with questions about distances, angles, and so on. We are bound to interpret the distance (δ) between the points $(a+bi, c+di)$ and $(a'+b'i, c'+d'i)$ as given by

$$\delta^2 = \{(a-a')+(b-b')i\}^2 + \{(c-c')+(d-d')i\}^2,$$

and there are corresponding theorems about angles.

On the whole, we think Prof. Hatton's book will be most useful in suggesting ways in which the Cartesian way of regarding complex points (and lines) is brought into line (without sacrificing logic) with the projective theory. What we may call the complex point (OP), meaning the involution (with a definite sense) of which O is the centre, and $-OP^2$ the invariant $(OQ.OQ' = -OP^2)$, is a perfectly definite idea, and is a special case of von Staudt's representation of any complex point in the harmonic form $(O \infty PP')$ or $(O \infty P'P)$ with the initial point O.

Von Staudt's theory is purely projective, apart from the discussion of improper casts. The Cartesian theory is bound to deal with metrical quantities, such as distance and angle, and simply because these notions are derivative, it offers a field of research of a more complicated character. It may be asserted with some confidence that any real extension of von Staudt's theory will be of a metrical kind, and that, if it is worth anything, it will be applicable to three dimensions (or more) as well as to two. G. B. MATHEWS.

P.S.—Since the above was written, I have had time to reflect further upon Prof. Hatton's book, and have read Prof. G. H. Hardy's review of it in a recent number of the *Mathematical Gazette*. I do not wholly agree with Prof. Hardy's attitude, because I still think that there are geometrical notions not reducible to arithmetic—still less to formal logic. But I do agree with him that Prof. Hatton's book has no theoretical value, and, disagreeable as it is, I think it is my duty to say so, especially as I have been informed that another reviewer has praised the book in absurdly exaggerated terms. G. B. M.

Motion Study and the Manual Worker.

Motion Study for the Handicapped. By Frank B. Gilbreth and Dr. Lillian Moller Gilbreth. (Efficiency Books.) Pp. xvi+165. (London: George Routledge and Sons, Ltd., 1920.) Price 8*s.* 6*d.* net.

THE work of Mr. Frank B. Gilbreth upon applied motion study and fatigue study is well known, and the present volume describes various extensions and additions to his previously recorded methods, especially with the intention of assisting men who are handicapped by the loss of a limb or of their eyesight. In Mr. Gilbreth's latest scheme the manual worker whose movements are being studied has a small electric light attached to the hand or other working member of the body, and thereby the path of the motions made can be determined in detail if a series of photographs is taken by kinematograph. Other photographs are taken with a stereoscopic camera, and by this means the path of the motion in three dimensions is ascertained. It is then possible to construct wire models showing exactly the path of a given motion, and such models are found to be very useful for instruction purposes. Series of models are exhibited at the Smithsonian Institution, Washington, and elsewhere, so that skilled mechanics are able to see for themselves what are considered to be the best methods of performing certain motions, and to determine if they themselves fall short of the ideal.

Again, Mr. Gilbreth represents on diagrams, termed "simultaneous-motion cycle charts," the results of his studies on micro-motion. Such charts, when read downwards, present in chronological sequence the various activities performed by any member of the body, the posture taken during the action, and the time consumed. If read across, the charts give a record of all the working members of the body at any one time, and they enable one to see which parts of the body are working most and which are being delayed. It is maintained that this chart system enables the workmen to visualise their efforts graphically, and thereby to lessen waste and increase efficiency.

The great ingenuity of Mr. Gilbreth's methods will be admitted by everyone, but it is more important for us to determine their practical value. Mr. Gilbreth photographs champions playing baseball, champion typists, skilled surgeons when operating, in addition to skilled tradesmen, and he believes that the skill shown is in every case based on one common set of fundamental principles, the principles of economy of effort and rhythm of motion. The application of this hypothesis to practical ends is, however, very far distant. The concrete instances quoted. of the employment of micro-motion study in actual practice are very few and not very striking, but doubtless it will take a good deal of time before they can be adequately tested and applied.

The portion of the book devoted specially to the handicapped describes several useful methods, though it strikes the uninitiated that they could have been evolved equally well without elaborate micro-motion study and motion-cycle charts. The one-armed typist is supplied with a typewriter

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